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ELASTOPLASTIC DEFORMATION AND THE LIMIT EQUILIBRIUM OF GRANULAR MEDIA†

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A theory of the deformation of a granular medium beyond the limit of elasticity is proposed, which takes into account both elastic and plastic deformations and enables the transition of the medium from a purely elastic state to a limit state to be traced. A system of functions is introduced which define the components of the stress and strain tensors, and a resolving system of equations of elastoplastic deformation of the medium is obtained for the case of a plane strained state. By taking the limit one can obtain the equations of the theory of limit equilibrium from these equations. The elastoplastic problem of the loading of a tube (a thick-walled circular cylinder) by internal pressure is solved. It is shown that the load corresponding to the transition of the tube to a purely plastic state over the whole cross section is not always identical with the load obtained using the limit equilibrium theory. The passage to the limit for which these two loads are the same is determined. The elastoplastic problem of the loading of a circular layer having a hole (a well) at the centre is solved in the new formulation. The depth of the layer for which a region of plastic deformation appears in the neighbourhood of the well are determined. The limit value of the depth at which the layer collapses is obtained. The residual stresses in the neighbourhood of the well when there is a repeated increase in the pressure to the value of the rock pressure are determined. The classical problem of the stressed state of a plane slope is solved in the elastoplastic formulation. A relation is established between the mechanical characteristics of the medium and the angle of the slope for which the slope always remains in the elastic state over the whole depth. It is shown that three different stress–strain states are possible in elastoplastic deformation. The transition to the limit state is traced and it is established that, unlike existing representations, plastic flow of the slope from a rigid plastic material is impossible for slope angles less than the angle of internal friction. For slope angles greater than the angle of internal friction plastic yield does not occur over the whole slope but only over its upper layer of a certain thickness.

According to established terminology, the mechanics of granular media means the science of the laws of deformation of soils, rocks, inherently granular and granulated media and other materials, the behaviour of which is related by the fact that the conditions under which a transition occurs from an elastic state to a state of plastic flow (the yield criterion) depends on the hydrostatic pressure.

Beginning with the basic work of Coulomb and up to the beginning of the 1960s, the mechanics of granular media developed mainly as the science of the statics of granular media. This was largely due to the fact that the equations describing the stressed state of a medium in the plane strained state [1], like the equations of the theory of perfect plasticity, are statically determinate, i.e. in the case of statically determinate boundary conditions they can be solved without having to invoke kinematic equations. The equations describing the stressed state are then hyperbolic type equations and have two families of characteristics, which are slip lines intersecting at angles which depend on the angle of internal friction of the granular medium.

Attempts to investigate the strained state of a granular medium are based, as a rule, on the assumption that the material behaves as a rigid plastic medium and is incompressible. The latter assumption meant that the equations for determining the velocity fields also have two families of characteristics, which turned out to be mutually orthogonal and, consequently, were not identical with the characteristics of the equations describing the stresses.

This contradiction was overcome using yield flow [2], based on applying the associated law to the yield criterion and the assumption of the rigid-plastic behaviour of the material. The main advantage

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of this theory is that the characteristics of the equations describing the stress and velocity field for the plane strained state are identical, and the regions of limit states can be defined uniquely. On the basis of this model a detailed investigation of the resolving system of equations and the lines of discontinuity of the velocities and stresses was carried out in [3–6], a number of new problems were solved, including those with mixed boundary conditions, and effective numerical methods of solving boundary-value problems were developed. The drawback of this theory, as also of any theory of limit equilibrium, is the fact that it only enables one to determine the limit loads and the distribution of the stresses and strains in plastic regions. This was combined historically in such a way that the theories mentioned above, following Coulomb, came to be called, somewhat inaccurately, theories of limit equilibrium. Their main objective is to find the limit loads, treated as loads, on reaching which, the medium passes to a limit state (a loss of stability occurs). Essentially, from the modern point of view, they are theories of the yield of an ideal rigid plastic material, while the limit loads are understood as the loads, on reaching which, such yield becomes possible. It is precisely in this sense that these terms will be used below.

An attempt to take into account the elastic properties of the material† using the well-known approach [2] leads to the result that the total strain rates begin to depend not only on the stresses but also on their partial time derivatives, which gives rise to considerable mathematical difficulties both when analysing the resolving system of equations, and when solving specific problems.

Another approach to determining the stress-strain state of a granular medium consists of using, as the equations relating the components of the stress and strain tensors, particularly in engineering applications [7, 8], relations similar to those of the theory of elasticity. In the initial states of the loading, the moduli of elasticity are assumed to be constant, i.e. the usual Hooke's law holds, and as the load increases they are assumed to be variable, determined from experiment. When processing experimental diagrams it is recommended that both shear and normal moduli should be used, i.e. the procedure employed is similar to the method in which variable moduli of elasticity are introduced in the method of elastic solutions developed in the deformation theory of plasticity. However, the hypotheses which enable this method to be used in the mechanics of granular media have not been formulated and a generally accepted deformation theory of granular media has not so far been developed. The transition from elastic state to a limit state has not been investigated to any great extent.

A similar situation also exists in the theory of plasticity. Thus, for the elastoplastic bending and twisting of rods and for the bending of plates the transition of the whole cross section of the rod or the plate to the plastic state occurs when there are external loads, identical with the loads given by the theory of limit equilibrium (see, for example, [9, 10]). On the other hand, for the elastoplastic deformation of a thick-walled tube the whole cross section of the tube transfers to a purely plastic state for a load less than the limit load, and in the case of bending and twisting the transition of the whole cross section to a purely plastic state is accompanied by an unlimited increase in the deformations, whereas in the case of a tube the deformations remain finite [11].

Before constructing a deformation theory of granular media, we note that in the theory of plasticity there is a theorem due to Il'yushin [12] which asserts that, in the case of simple loading, yield theory and deformation theory lead to the same results, that is, the equations of yield theory can be integrated. To prove this theorem the material is assumed to be incompressible and a power law of hardening is used. Since a change in volume occurs in the model used in [2], while the material is assumed to be non-hardening, Il'yushin's theorem cannot be applied directly to it. It was shown in [13, 14] that for the case of simple deformation the equations of yield theory [2] can be integrated and the governing equations of the mechanics of granular media can be represented in the form of the equations of the deformation theory of plasticity [15].

1. THE CONDITION FOR THE EQUATIONS OF YIELD THEORY TO BE INTEGRABLE

In the mechanics of soils, rock and granular media, the criterion for the limit state (the condition for plastic yield) is usually taken in the form

$$\Phi(\sigma, \tau) = 0, \quad \sigma = \sigma_{ii}/3, \quad \tau = \sqrt{s_{ij}s_{ij}/2} \tag{1.1}$$

where $s_{ij} = \sigma_{ij} - \sigma\delta_{ij}$ are the components of the stress deviator. The form of the function is found by experiment, where $d\tau/d\sigma \leq 0$. Here and henceforth we will assume that the tensile stresses are positive.

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By the associated flow law [2], for the components of the plastic strain rates and rates of change of the relative volume, we will have

$$\dot{\varepsilon}_{ij}^{p} = \lambda\frac{\partial\Phi}{\partial\sigma_{ij}} = \lambda\left(\frac{1}{3}\frac{\partial\Phi}{\partial\sigma}\delta_{ij} + \frac{\partial\Phi}{\partial\tau}\frac{s_{ij}}{2\tau}\right), \quad \dot{\varepsilon}^{p} = \frac{\lambda}{3}\frac{\partial\Phi}{\partial\sigma} \tag{1.2}$$

We will introduce the deviator of the plastic strain rates with components $\dot{e}_{ij}^{p} = \dot{\varepsilon}_{ij}^{p} - \dot{\varepsilon}^{p}\delta_{ij}$ and the intensity of the shear plastic strain rates $\eta^{p} = \sqrt{\dot{e}_{ij}^{p}\dot{e}_{ij}^{p}/2}$. Then for the parameter λ we obtain $\lambda = 2\eta^{p}/|\partial\Phi/\partial\tau|$. Without loss of generality we will assume that $\partial\Phi/\partial\tau > 0$. Then relations (1.2) take the form

$$\dot{e}_{ij}^{p} = \frac{\eta^{p}}{\tau}s_{ij}, \quad \dot{\varepsilon}^{p} = -\frac{2}{3}\eta^{p}\frac{d\tau}{d\sigma} \tag{1.3}$$

In the case of the simple deformation [13, 14] $\varepsilon_{ij}^{p} = \varepsilon_{ij}^{0}t$, and consequently $\dot{\varepsilon}_{ij}^{p} = \varepsilon_{ij}^{0}$, $\dot{e}_{ij}^{p} = e_{ij}^{0}$, $\eta^{p} = \sqrt{e_{ij}^{0}e_{ij}^{0}/2}$, i.e. the strain rates do not change with time, in the hyperspace ε_{ij}^{p} the deformation trajectory is a straight line, while the components of the stress tensor, by virtue of relations (1.1) and (1.3), remain constant. The quantities ε_{ij}^{0} must be chosen so that the condition that the deformation trajectory is orthogonal to the limit surface (1.1) is satisfied, i.e. the second of equalities (1.3) is satisfied. Then

$$\varepsilon^{0} = -\sqrt{2e_{ij}^{0}e_{ij}^{0}}\,d\tau/d\sigma$$

The important fact was not taken into account in [13, 14].

Integrating the equations with respect to t and taking into account the fact that

$$\dot{\varepsilon}^{p}t = \varepsilon^{0}t = \varepsilon^{p}, \quad \eta^{p}t = \sqrt{e_{ij}^{0}e_{ij}^{0}/2}\,t = \sqrt{e_{ij}^{p}e_{ij}^{p}/2} = \gamma^{p}$$

where γ^{p} is the intensity of shear plastic strains, we obtain the expressions

$$e_{ij}^{p} = \frac{\gamma^{p}}{\tau}s_{ij} + e_{ij}^{*}, \quad \varepsilon^{p} = -\frac{2}{3}\gamma^{p}\frac{d\tau}{d\sigma} + \varepsilon^{*} \tag{1.4}$$

Here ε^{*} and e_{ij}^{*} are constants of integration, which must be determined from the initial conditions. If we assume that the time t is measured from the instant of the onset of the plastic state, the constants ε^{*} and e_{ij}^{*} must be taken equal to zero.

In the theory of elasticity the strain and stress tensors are related by Hooke's law

$$e_{ij}^{e} = \frac{1}{2G}s_{ij}, \quad \varepsilon^{e} = \frac{1}{2G}\frac{1-2\nu}{1+\nu}\sigma, \quad \gamma^{e} = \frac{\tau}{2G} \tag{1.5}$$

where G is the shear modulus and ν is Poisson's ratio.

Assuming, as usual [15], that the total deformations can be represented in the form of the sum of the plastic and elastic deformations, from Eqs (1.4) and (1.5) we obtain the relations

$$e_{ij} = e_{ij}^{e} + e_{ij}^{p} = \left(\frac{1}{2G} + \frac{\gamma^{p}}{\tau}\right)s_{ij} = \frac{\gamma}{\tau}s_{ij}, \quad \varepsilon = \varepsilon^{e} + \varepsilon^{p} = \left(\frac{1}{2G}\frac{1-2\nu}{1+\nu} - \frac{2}{3}\frac{\gamma^{p}}{\sigma}\frac{d\tau}{d\sigma}\right)\sigma \tag{1.6}$$

similar to the Hencky relations in the deformation theory of plasticity [15]. Note that they were obtained from other considerations previously in [13, 14]. This representation of the governing equations enables iteration methods to be developed for solving the problems of the mechanics of granular media [16, 17], similar to the methods of elastic solutions in the deformation theory of plasticity.

If relation (1.1) is a linear function of the form

$$\Phi(\sigma, \tau) = \tau - (H - \sigma)\operatorname{tg}\alpha = 0 \tag{1.7}$$

where H and α are constants of the material, which determine its mechanical properties, then in the space of the principal stresses σ_1, σ_2 and σ_3 it represents a circular cone with axis $\sigma_1 = \sigma_2 = \sigma_3$ and angle at the vertex $\varphi = \operatorname{arctg}(\sqrt{2/3}\operatorname{tg}\alpha)$.

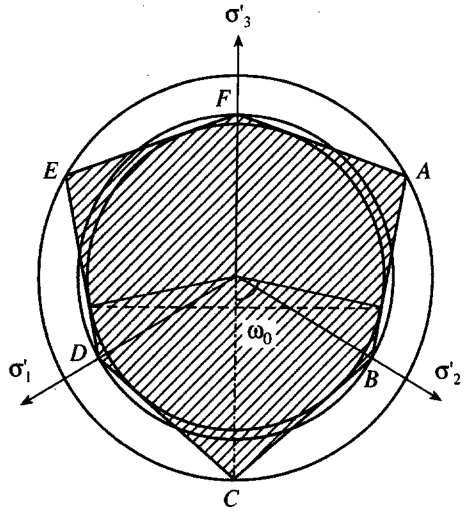

Fig. 1

The Coulomb criterion usually used in applications [1, 18] (if one gives up the requirement that $\sigma_1 \geq \sigma_2 \geq \sigma_3$) has the form

$$|\sigma_1 - \sigma_2| \leq [2H - (\sigma_1 + \sigma_2)]\sin\rho \qquad (123) \tag{1.8}$$

where ρ is the angle of internal friction and H is the reduced coefficient of cohesion, and in this space is a hexagonal pyramid with three planes of symmetry, which intersect the same axis $\sigma_1 = \sigma_2 = \sigma_3$.

The use of criterion (1.8) when investigating the equations describing the three-dimensional stressed state of a medium leads to considerable mathematical difficulties, and, from this point of view, the cone (1.7) can be regarded as an approximation of criterion (1.8).

Usually [18] it is recommended that a cone inscribed in the pyramid (1.8) should be used as an approximation. However, other approaches are also possible. In Fig. 1 we show a hexagon formed by the intersection of the pyramid (1.8) with the deviator plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$, and three circles formed by the intersection of the cone (1.7) with the same plane, for three different values of α

$$\operatorname{tg}\alpha_1 = \frac{\sqrt{3}\sin\rho}{\sqrt{3 + \sin^2\rho}}, \quad \operatorname{tg}\alpha_{2,3} = \frac{2\sqrt{3}\sin\rho}{3 \pm \sin\rho} \tag{1.9}$$

where α_1 corresponds to the cone inscribed in a pyramid (1.7), α_2 corresponds to the small cone circumscribed about it and α_3 corresponds to the large cone circumscribed about it.

2. THE RESOLVING SYSTEM OF EQUATIONS. THE PLANE STRAINED STATE

The principal values of the stress tensor can be expressed in terms of the quantities α and τ and a new function ω in the form

$$\sigma_{1,2} = \sigma + 2\tau\cos(\omega \mp \pi/3)/\sqrt{3}, \quad \sigma_3 = \sigma - 2\tau\cos\omega/\sqrt{3} \tag{2.1}$$

Then the principal values of the strain tensor, on the basis of expressions (1.6), can be written as

$$\varepsilon_{1,2} = \varepsilon + 2\gamma\cos(\omega \mp \pi/3)/\sqrt{3}, \quad \varepsilon_3 = \varepsilon - 2\gamma\cos\omega/\sqrt{3} \tag{2.2}$$

We then have the finite relation

$$\varepsilon = \frac{1}{2G}\frac{1-2\nu}{1+\nu}\sigma - \frac{2}{3}\left(\gamma - \frac{\tau}{2G}\right)\frac{d\tau}{d\sigma} \tag{2.3}$$

The components of the stress tensor in an orthogonal Cartesian system of coordinates can be represented in the form

$$\sigma_{ij} = \sigma\delta_{ij} + 2\tau[n_{i1}n_{j1}\cos(\omega - \pi/3) + n_{i2}n_{j2}\cos(\omega + \pi/3) - n_{i3}n_{j3}\cos\omega]/\sqrt{3} \tag{2.4}$$

where n_{ij} are nine direction cosines, specifying the directions of the principal axes of the stress tensor, uniquely defined by the three Euler Angles. The components of the strain tensor can be represented in a similar way in terms of the quantities (2.2) and (2.3) and the same values of the direction cosines. Consequently, the components of the stress and strain tensors are defined by the five functions σ, τ, ε, γ, ω and the three Euler angles. Three differential equations of equilibrium and the two finite relations (1.1) and (2.3) are insufficient to find them, i.e. the problem is statically indeterminate and one must invoke the equations for the displacements to solve it.

We will obtain the resolving system of differential equations for the case of a plane strained state when $\varepsilon_z = 0$ and the stress and strain tensors are independent for the z coordinate. The z axis is the principal axis, i.e. $\sigma_z = \sigma_3$; the angle of proper rotation φ is the angle between the x axis and the direction of the principal stress σ_1, while the remaining Euler angles are equal to zero. Then Eq. (2.4) can be represented in the form

$$\sigma_{x,y} = \sigma + \tau[\cos\omega/\sqrt{3} \pm \sin\omega\cos 2\varphi], \quad \tau_{xy} = \tau\sin\omega\sin 2\varphi \tag{2.5}$$

The principal values of the components of the strain tensor (2.2) can be rewritten as

$$\varepsilon_{1,2} = 2\gamma\sin(\pi/3 \pm \omega), \quad \varepsilon_3 = 0 \tag{2.6}$$

while, for the components of the strain tensor themselves, we obtain

$$\varepsilon_{x,y} = \gamma[\sqrt{3}\cos\omega \pm \sin\omega\cos 2\varphi], \quad \gamma_{xy} = \gamma\sin\omega\sin 2\varphi \tag{2.7}$$

Substituting the components of (2.5) into the two equilibrium equations

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0, \quad \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} = 0$$

we obtain

$$\begin{aligned} &\frac{\partial}{\partial x}[\sigma + \tau(\cos\omega/\sqrt{3} + \sin\omega\cos 2\varphi)] + \frac{\partial}{\partial y}(\tau\sin\omega\sin 2\varphi) = 0 \\ &\frac{\partial}{\partial y}[\sigma + \tau(\cos\omega/\sqrt{3} - \sin\omega\cos 2\varphi)] + \frac{\partial}{\partial x}(\tau\sin\omega\sin 2\varphi) = 0 \end{aligned} \tag{2.8}$$

The two equations (2.8) and criteria (1.1) are insufficient to determine the stressed state, i.e. the problem continues to remain statically indeterminate.

Expressing the components of the strain tensor on the left-hand sides of formulae (2.7) in terms of the components of the displacement vector

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

and dividing the first two equations by the third, we obtain two equations for determining the displacement field

$$\begin{aligned} &2\sin\omega\sin 2\varphi\frac{\partial u}{\partial x} - (\sqrt{3}\cos\omega + \sin\omega\cos 2\varphi)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \\ &2\sin\omega\sin 2\varphi\frac{\partial v}{\partial y} - (\sqrt{3}\cos\omega - \sin\omega\cos 2\varphi)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \end{aligned} \tag{2.9}$$

The finite relation of (2.3) in the case of a plane strained state takes the form

$$\frac{2}{\sqrt{3}}\gamma\cos\omega = \frac{1}{2G}\frac{1-2\nu}{1+\nu}\sigma - \frac{2}{3}\left(\gamma - \frac{\tau}{2G}\right)\frac{d\tau}{d\sigma} \tag{2.10}$$

The four differential relations (2.8) and (2.9) and two finite relations (1.1) and (2.10) contain six required functions σ, τ, φ, ω, u, υ and enable us to obtain the components of the stress tensor and of the displacement vector, u and υ.

The system of equations obtained can be simplified considerably if we neglect the elastic deformations. To do this it is sufficient in formula (2.10) to put the terms containing the factor $1/G$ equal to zero. Assuming, to simplify further calculations, that the criterion of the limit state (1.1) has the form of the linear function (1.7), we obtain from (2.10)

$$\omega = \arccos\frac{\operatorname{tg}\alpha}{\sqrt{3}} = \omega_0 \tag{2.11}$$

Note that the same result can be obtained if we assume that the deformations of the medium and, consequently, the quantity γ, increase without limit. As will be shown below, when solving specific problems, it is precisely when we take the limit $\gamma \to \infty$ that the solutions obtained using the theory of elastoplastic deformation tend to the solutions given by the theory of limit equilibrium.

When deriving formula (2.11) we assumed that the cone (1.7) is inscribed in a Coulomb hexagonal pyramid. In that case the lines of contact of the cone with the pyramid lie in the plane

$$\sigma_3 = -H\sin^2\rho + (1+\sin^2\rho)(\sigma_1+\sigma_2)/2 \tag{2.12}$$

which passes through the vertex of the pyramid and intersects the deviator plane along the straight line shown dashed in Fig. 1. The normal to the surface of the cone is orthogonal to the $\sigma_3 = \sigma_z$ axis only along these two contact lines and, consequently, only along them is the condition of plane strain $\varepsilon_3 = \varepsilon_z = 0$ satisfied for a rigid-plastic material. In other words, in this case, in the space of the principal stresses the points corresponding to this stressed state can only lie on these contact lines. Then it is obvious that criterion (1.7) can be replaced by condition (1.8), which in this case can be rewritten as

$$t = (H-s)\sin\rho; \quad s = (\sigma_1+\sigma_2)/2, \quad t = (\sigma_1-\sigma_2)/2 \tag{2.13}$$

Using the well-known formulae $\sigma_{x,y} = s \pm t\cos 2\varphi$, $\tau_{xy} = t\sin 2\varphi$, we obtain from condition (2.13)

$$\sigma_{x,y} = s \pm (H-s)\sin\rho\cos 2\varphi, \quad \tau_{xy} = (H-s)\sin\rho\sin 2\varphi$$

Note that, taking formulae (2.11), (2.12) and (1.7) into account, these expressions can be obtained directly from relations (2.5). In this case Eqs (2.8) take the form

$$\begin{aligned}
&\frac{\partial s}{\partial x} + \cos 2\varphi\frac{\partial t}{\partial x} + \sin 2\varphi\frac{\partial t}{\partial y} - 2t\left(\sin 2\varphi\frac{\partial \varphi}{\partial x} - \cos 2\varphi\frac{\partial \varphi}{\partial y}\right) = 0\\
&\frac{\partial s}{\partial y} + \sin 2\varphi\frac{\partial t}{\partial x} - \cos 2\varphi\frac{\partial t}{\partial y} + 2t\left(\cos 2\varphi\frac{\partial \varphi}{\partial x} + \sin 2\varphi\frac{\partial \varphi}{\partial y}\right) = 0
\end{aligned} \tag{2.14}$$

Taking (2.13) into account for the boundary conditions, specified in stresses, they enable us to determine the stressed state of the medium, i.e. as a result of neglecting the elastic deformations the problem has become statically determinate.

Equations (2.14) are hyperbolic-type equations and have two families of characteristics

$$\lambda \mp \varphi = \text{const}, \quad dy/dx = \operatorname{tg}(\varphi \mp \psi) \tag{2.15}$$

Here we have introduced the notation

$$dt/ds = \cos 2\varphi, \quad 2\psi = \pi/2 + \rho, \quad d\lambda = \sin 2\psi\, ds/t$$

The equations can be rewritten as

$$2\sin 2\varphi\frac{\partial u}{\partial x} - (\cos 2\varphi - \cos 2\psi)\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right) = 0$$
$$2\sin 2\varphi\frac{\partial \upsilon}{\partial y} + (\cos 2\varphi + \cos 2\psi)\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right) = 0 \qquad (2.16)$$

They are also hyperbolic-type equations, and the equations of their characteristics are identical with (2.15), while the conditions along the characteristics have the form

$$du/d\upsilon = -\mathrm{tg}(\varphi \mp \psi) \qquad (2.17)$$

Note that Eqs (2.14)–(2.17), are identical, apart from the notation, with the equations of the theory of limit equilibrium, obtained in [19] using criterion (1.1), the associated flow law (1.2) and the assumption of rigid-plastic behaviour of the material. The difference is the fact that in Eqs (2.16) u and υ are the components of the displacement vector, while previously in [19] u and υ were the components of the velocity vector. However, Eqs (2.14) and (2.16) are also the equations of the theory of limit equilibrium, but have been obtained by taking the limit from Eqs (2.8) and (2.9), taking into account the elastic deformations in the plastic regions. Consequently, the theory of elastoplastic deformation developed above includes the theory of limit equilibrium and enables one to investigated the transition of a medium from the purely elastic state to a limit state.

3. THE LOADING OF A CYLINDRICAL TUBE BY INTERNAL PRESSURE

In order to apply the equations obtained, we will consider the problem of the elastoplastic deformation of a cylindrical tube of inner radius a and outer radius b, loaded by an internal pressure p and which is in a plane strained state. The external pressure is equal to zero.

So long as the pressure p is sufficiently small, the tube is in an elastic state over the whole cross section. The components of the strain tensor in a cylindrical system of coordinates connected with the axis of the tube are expressed in terms of the radial displacement as follows:

$$\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\theta = \frac{u}{r}, \quad \varepsilon_z = 0 \qquad (3.1)$$

They satisfy the equation of compatibility of the deformations

$$\frac{d\varepsilon_\theta}{dr} + \frac{\varepsilon_\theta - \varepsilon_r}{r} = 0 \qquad (3.2)$$

and are related to the components of the stress tensor σ_r and σ_θ by Hooke's law

$$\sigma_{r,\theta} = \frac{2G}{1-2\nu}((1-\nu)\varepsilon_{r,\theta} + \nu\varepsilon_{\theta,r}), \quad \sigma_z = \nu(\sigma_r + \sigma_\theta) \qquad (3.3)$$

The latter, in turn, satisfy the equilibrium equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (3.4)$$

Solving system of equations (3.1)–(3.4), we obtain

$$\sigma_{r,\theta} = C_2 \mp \frac{C_1}{r^2}, \quad \sigma_z = 2\nu C_2, \quad u = \frac{1}{2G}\left((1-2\nu)C_2 r + \frac{C_1}{r}\right) \qquad (3.5)$$

The constants of integration can be obtained from the boundary conditions

$$\sigma_r|_{r=a} = -p, \quad \sigma_r|_{r=b} = 0 \qquad (3.6)$$

Finally, the solution takes the form

$$\sigma_{r,\theta} = \chi p\left(1 \mp \frac{b^2}{r^2}\right), \quad \sigma_z = 2\nu\chi p, \quad u = \frac{1}{2G}\chi p\left((1-2\nu)r + \frac{b^2}{r}\right); \quad \chi = \frac{a^2}{b^2 - a^2} \tag{3.7}$$

It holds only in the case when the tube is in an elastic state over the whole cross-section. In order to obtain the range in which it is applicable, we substitute expression (3.7) for $r = a$ into the yield criterion (1.7), and we obtain the critical pressure

$$p_0 = \frac{H\operatorname{tg}\alpha}{\sqrt{(1+\chi)^2 + (1-2\nu)^2\chi^2/3} + 2(1+\nu)\chi\operatorname{tg}\alpha/3} \tag{3.8}$$

after exceeding which in the tube a region $a \le r \le c$ of plastic state appears. In the region $c \le r \le b$, as before, formulae (3.5) remain valid, except that the constants of integration must be found from the condition for σ_r to be continuous at the point c and from the second of boundary conditions (3.6).

Since the r, θ, z axes are the principal axes of the stress and strain tensors, the components of these tensors are identical with expressions (2.1) and (2.2), apart from the replacement of the subscripts 1, 2 and 3 by θ, r and z. Hence, the non-zero components of the stress and strain tensors have the form

$$\sigma_{r,\theta} = \sigma + 2\tau\cos(\omega \pm \pi/2)/\sqrt{3}, \quad \sigma_z = \sigma - 2\tau\cos\omega/\sqrt{3} \tag{3.9}$$

$$\varepsilon_{r,\theta} = \varepsilon + 2\gamma\cos(\omega \pm \pi/3)/\sqrt{3}, \quad \varepsilon_z = \varepsilon - 2\gamma\cos\omega/\sqrt{3} = 0 \tag{3.10}$$

Introducing expressions (3.9) and (3.10) into the equilibrium equation (3.4) and into the equation of compatibility of the deformations (3.2) and using the yield criterion in the form (1.7), we obtain

$$r\frac{d}{dr}\left[\sigma + \frac{2}{\sqrt{3}}(H-\sigma)\operatorname{tg}\alpha\cos\left(\omega + \frac{\pi}{3}\right)\right] - 2(H-\sigma)\operatorname{tg}\alpha\sin\omega = 0 \tag{3.11}$$

$$r\frac{d}{dr}\left[\gamma\sin\left(\omega + \frac{\pi}{3}\right)\right] + \gamma\sin\omega = 0 \tag{3.12}$$

Taking expression (1.7) into account, the finite relation (2.10) takes the form

$$\frac{2}{\sqrt{3}}\gamma\cos\omega = \frac{1}{2G}\frac{1-2\nu}{1+\nu}\sigma + \frac{2}{3}\operatorname{tg}\alpha\left(\gamma - \frac{1}{2G}(H-\sigma)\operatorname{tg}\alpha\right) \tag{3.13}$$

The system of three equations (3.11)–(3.13) enables us to obtain the three required functions σ, γ and ω, which completely define the stress–strain state in the region $a \le r \le c$. The radial displacement can be found from the formula

$$u = \varepsilon_\theta r = 2r\gamma\sin(\omega + \pi/3)$$

The boundary conditions for Eqs (3.11) and (3.12) are found from the condition of continuity of the components of the stress and strain tensors at the boundary $r = c$ between the elastic-state and the plastic-state regions. After conversion, these conditions take the form

$$r = c\colon \omega = \omega_c, \quad \sigma = \sigma_c, \quad \gamma = \gamma_c \tag{3.14}$$

where

$$\operatorname{tg}\omega_c = \frac{\sqrt{3}}{1-2\nu}\frac{b^2}{c^2}, \quad \sigma_c = H\left[1 + \frac{\sqrt{3}}{2}\frac{1-2\nu}{1+\nu}\frac{1}{\operatorname{tg}\alpha\cos\omega_c}\right]^{-1}, \quad \gamma_c = \frac{1}{2G}(H-\sigma_c)\operatorname{tg}\alpha \tag{3.15}$$

In the elastic-state region $c \le r \le b$, as before, formulae (3.5) hold, where the second of conditions (3.6) enables us to write the first three of them as

$$\sigma_{r,\theta} = C_1\left(\frac{1}{b^2} \mp \frac{1}{r^2}\right), \quad \sigma_z = 2\nu\frac{C_1}{b^2}; \quad \frac{C_1}{b^2} = \frac{3\sigma_c}{2(1+\nu)} \tag{3.16}$$

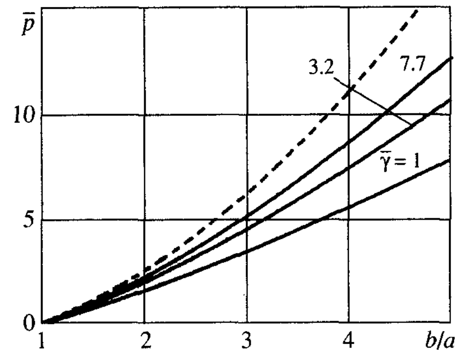

Fig. 2

The quantity σ_c is given by the second expression in (3.15).

Assuming the quantity c/b to be a loading parameter, we obtain a Cauchy problem for Eqs (3.11)–(3.13) with initial conditions (3.15) at the point c and a region of integration $a \leq r \leq c$; in the elastic-state region $c \leq r \leq b$ the solution is given analytically by relations (3.16). The internal pressure p corresponding to a specific value of the parameter c/b, will be obtained from the condition $p = -\sigma_r(a)$.

Although this system of equations can also be solved analytically, this is inconvenient in view of the complexity of the expressions obtained. A detailed numerical solution of the problem was obtained in [20]. We will merely note that when the loading parameter c/b changes from a value of a/b to 1, the internal pressure p varies from p_0 (3.8) to a certain pressure p_1, corresponding to the transition of the tube to the plastic state over the whole cross section. However, the deformations of the tube remain finite.

When the internal pressure p in the tube exceeds p_1, there is no elastic-state region and the tube is completely in a plastic state (but its supporting power is not exhausted). Equations (3.11)–(3.13) with boundary conditions (3.6) hold as before over the whole cross section of the tube. Since in this case the first of conditions (3.14) and (3.15) loses its meaning, we will introduce the value of γ on the external contour of the tube as the loading parameter. The specified value of $\gamma(b)$ and also the second and third of conditions (3.14) and (3.15) enable us to obtain the values of $\sigma(b)$ and $\omega(b)$ and thereby determine the Cauchy problem in the region $a \leq r \leq b$.

The internal pressure p corresponding to the chosen value of $\gamma(b)$ will be obtained, as before, from the condition $p = -\sigma_r(a)$. Normalizing the function $\gamma(b)$, we introduce the dimensionless parameter $\bar{\gamma} = \bar{\gamma}(b)$ so that $\bar{\gamma}(b) = 1$ corresponds to the transition of the tube to the plastic state over the whole cross section. In Fig. 2 we show curves of the dimensionless quantity $\bar{p} = p(b/a)/H$ for $\mathrm{tg}\alpha = 1.7$ and $\nu = 0.1$ and different values of $\bar{\gamma}$. As $\bar{\gamma}$ increases, which corresponds to an increase in the displacement $u(b)$ on the external contour of the tube, the curves in Fig. 2 approach the limit curve, shown dashed in the figure. The form of this curve can be obtained fairly simply if we consider the limiting case when the loading of the tube leads to infinitely large deformations. As was shown above, such a passage to the limit is equivalent to putting $1/G = 0$ in the equations of elastoplastic deformation and converting them into the limit-equilibrium equations.

In this case we obtain $\omega = \omega_0 = \mathrm{const}$ (2.11), and Eqs (3.11) and (3.12) can be fairly simply integrated:

$$\sigma = H - Ar^{-m}, \quad \gamma = Br^{-n}, \quad m, n = 2/[1 + (\mathrm{ctg}^2\alpha - 1/3)^{\pm 1/2}] \tag{3.17}$$

where A and B are constants of integration. Boundary conditions (3.6) enable us to determine the value of A, and also the internal pressure corresponding to this limit solution

$$p^* = H((b/a)^m - 1) \tag{3.18}$$

The constant B can be chosen arbitrarily, since the material is unstrengthened, i.e. in the limit state the displacements in the tube can only be determined to within an arbitrary constant factor.

Thus, depending on the value of the internal pressure p the tube can be in one of three states of deformation: purely elastic when $0 \leq p \leq p_0$, elastoplastic when $p_0 < p \leq p_1$ and purely plastic when $p_1 < p < p^*$. When $p = p^*$ the deformations of the tube, obtained from the elastoplastic solution, becomes infinitely large (fracture occurs). Internal pressures exceeding p^* are impossible. If follows from these results that the transition of the tube to the plastic state over the whole cross section does

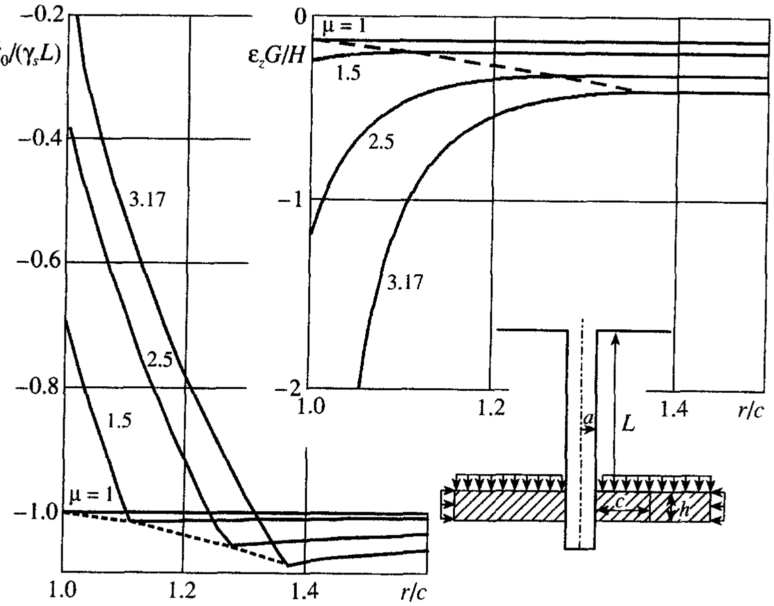

Fig. 3

not denote the onset of limit state. The loads p asymptotically approach the limit loads p^* (the dashed curve in Fig. 2) when the deformations of the tube increase without limit. Note that this was not pointed out earlier in [20].

In the majority of cases the values of p_1 and p^* differ considerably, while when Poisson's ratio ν and the angle of internal friction ρ satisfy the equality $1 - 2\nu = \sin\rho$, they are simply identical (here the transition from α to ρ corresponds to the first formula of (1.9)). This equality is satisfied, in particular, when the material of the tube is incompressible ($\nu = 1/2$) and ideal plastic ($\sin\rho = 0$).

The problem of the limit state of a cylindrical tube was considered by Nadai [18], assuming that $\sigma_z = (\sigma_r + \sigma_\theta)/2 = \sigma$ and that the material is incompressible. The components of the stress tensor σ_r and σ_θ and the value of the limit pressure he obtained are identical with the results obtained here for a cone inscribed in the Coulomb pyramid, i.e. $\mathrm{tg}\alpha = \mathrm{tg}\alpha_1$, where the dependence of $\mathrm{tg}\alpha$ on the angle of internal friction ρ is given by the first formula of (1.9). However, the component σ_z and the displacement u differ considerably from those obtained above. Nadai did not consider the elastoplastic state of the tube and the transition to a limit state.

4. THE ELASTOPLASTIC DEFORMATION OF A CIRCULAR STRATUM

We will consider the following problem as the second example. Under a layer of height L of ground with a specific gravity γ_s there is a stratum of material of thickness h, where $h \ll L$. A well of radius a is bored into the ground and the stratum and is filled with a liquid with specific gravity γ_l (see Fig. 3). It is required to determine the stress-strain state in the stratum, assuming that the transition from the elastic state to the limit state is determined by criterion (1.7).

If there is no well, the stressed state in the stratum corresponds to hydrostatic compression: $\sigma_{r,\theta,z} = -\gamma_s L$, i.e. the stresses are equal to the rock pressure. When there is a well the elastic stresses are expressed by formulae (3.5), and the constants C_1 and C_2 are found from the boundary conditions

$$\sigma_r|_{r=a} = -\gamma_l L, \quad \sigma_r, \sigma_\theta|_{r\to\infty} \to -\gamma_s L$$

The components of the stress tensor take the form

$$\sigma_{r,\theta} = -\gamma_s L(1 \mp (1-\Delta)(a/r)^2) = \sigma \pm \tau, \quad \sigma_z = -\gamma_s L; \quad \Delta = \gamma_l/\gamma_s \tag{4.1}$$

From Hooke's law we obtain

$$\varepsilon_z = -\frac{1}{2G}\frac{1-2\nu}{1+\nu}\gamma_s L, \quad u = \varepsilon_\theta r = -\frac{1}{2G}\gamma_s L\left[\frac{1-2\nu}{1+\nu}r + (1-\Delta)\frac{a^2}{r}\right] \tag{4.2}$$

Hence, for elastic deformation the presence of a well gives rise to no additional displacements in the vertical direction.

Formulae (4.1) and (4.2) hold so long as the whole stratum is in the elastic state. Substituting expression (4.1) into criterion (1.7) with $r = a$, we obtain the critical value of the depth at which the stratum lies

$$L_1 = \frac{H}{\gamma_s}\frac{\operatorname{tg}\alpha}{1-\Delta-\operatorname{tg}\alpha} \tag{4.3}$$

after exceeding which a region of plasticity $a \le r \le c$ occurs.

Replacing the subscripts 1, 2, 3 $\to \theta, r, z$ in formulae (2.1) and (2.2), we obtain

$$-\gamma_s L = \sigma - 2(H-\sigma)\operatorname{tg}\alpha\cos\omega/\sqrt{3} \quad (=\sigma_z) \tag{4.4}$$

The finite relation (2.3) takes the form

$$\varepsilon = \frac{1}{2G}\frac{1-2\nu}{1+\nu}\sigma + \frac{2}{3}\operatorname{tg}\alpha\left(\gamma - \frac{1}{2G}(H-\sigma)\operatorname{tg}\alpha\right) \tag{4.5}$$

Substituting the expressions for the components of the stress and strain tensors into Eqs (3.2) and (3.4), we obtain Eq. (3.11) and the equation

$$r\frac{d}{dr}[\varepsilon + 2\gamma\cos(\omega-\pi/3)/\sqrt{3}] + 2\gamma\sin\omega = 0 \tag{4.6}$$

The two differential equations ((3.11) and (4.6)) and the two algebraic equations ((4.4) and (4.5)) enable us to obtain the four required functions σ, ω, ε and γ, which completely define the stress–strain state in the region $a \le r \le c$.

In the elastic-state region $c \le r < \infty$ the components of the stress tensor, as before, will have form (3.5); taking into account the conditions at infinity we have $C_2 = -\gamma_s L$, and then

$$\sigma_{r,\theta} = -\gamma_s L \pm C_1/r^2 \tag{4.7}$$

From the conditions of continuity of the components of the stress tensor on the boundary $r = c$ between the regions of the elastic state and the limit state, we have boundary conditions (3.14) in which

$$\omega_c = -\pi/2; \quad \sigma_c = -\gamma_s L, \quad \gamma_c = \frac{1}{2G}(H-\sigma_c)\operatorname{tg}\alpha \tag{4.8}$$

and we obtain $C_1 = c^2(H + \gamma_s L)\operatorname{tg}\alpha$ for the constant in formula (4.7).

If we introduce the relative radius r/c, then conditions (3.14) and (4.8) for $r/c = 1$ and the specified values of γ_s and $L > L_1$ define the Cauchy problem for the system of equations (3.11), (4.4)–(4.6). Their numerical integration is carried out up to the point $a/c < 1$, at which the following boundary condition is satisfied

$$\sigma_a + 2(H-\sigma_a)\operatorname{tg}\alpha\cos(\omega_a + \pi/3)/\sqrt{3} = -\gamma_l L \quad (=\sigma_r|_{r=a}) \tag{4.9}$$

Detailed results of such calculations were presented in [20]. However, the system of equations (4.4), (4.9) can be solved in explicit form and show that a solution only exists when

$$L \le L_2 = 2\frac{H}{\gamma_s}\frac{\sin\rho}{(1-\Delta)-(1+\Delta)\sin\rho}$$

It turns out that when $L \to L_2$ the components of the strain tensor increase without limit, while when $L > L_2$ there is no solution which corresponds physically to fracture (collapse) of the stratum. This interesting fact is illustrated by a number of $\varepsilon_z G/H$ curves for a number of values of the parameter $\mu = L/L_1$, shown in the right upper part of Fig. 3. As previously [20], we take $\operatorname{tg}\alpha = \nu = \Delta = 1/3$ for the mechanical characteristics of the material, in which case the limit value of the parameter $\mu = L_2/L_1 = 3.178$. The intersection of the dashed line in Fig. 3 with the curve corresponding to a certain value of the parameter μ gives the dimensionless value of the boundary of the elastic-state region and the region of plasticity r/c for this value of the parameter μ.

Thus, if a well is bored into the stratum, then, when condition (4.3) is satisfied, a region of plastic yield will occur in it. If we then increase the pressure in the well, the stratum will gradually become overloaded. When the pressure becomes equal to the rock pressure, unlike the case of purely elastic deformation of the stratum, it does not return to the initial state of hydrostatic compression, and residual strains occur in it. In Fig. 3 we show the results of calculation of the residual stress $\tilde{\sigma}_\theta$ in the stratum referred to the value of the rock pressure $\gamma_s L$, for a number of values of μ. The case $\mu = 1$ corresponds to the horizontal straight line (purely elastic deformation). The dashed line is the boundary between the regions of elasticity and plasticity. From these curves we can judge how close the pressure in the stratum is to the rock pressure after overloading, i.e. how much the predictions of the elastic and elastoplastic theories differ. Calculations show that the residual radial stress in the stratum differs from the rock pressure by no more than 10%, while the residual stress $\tilde{\sigma}_\theta$ turns out to be considerably less than the rock pressure in modulus – the difference approaches 90% (for other values of the parameters of the problem, for example, for a lower ratio of the densities Δ, the difference may even exceed 100%, i.e. the residual stress $\tilde{\sigma}_\theta$ becomes a tensile stress).

This result is of considerable practical importance when constructing a theory of the hydrofracture of the stratum. In existing research on hydrofracture [21, 22] it is assumed that the stratum is in an elastic state. The values of the hydrofracture pressure then obtained from theoretical calculations turn out to be greater than those observed in practice, and these are almost everywhere less than the rock pressure $\gamma_s L$. This difference has been explained by the scouring of argillanceous formations, which bound the productive stratum above and below, during the drilling process, the smaller values of the side pressure compared with the rock pressure, etc. It is easy to see that when the residual stresses in the neighbourhood of the well are taken into account the same result is obtained. Moreover, what is more, for the rocks encountered in practice, the value of L_1 is fairly small, of the order of hundreds of meters.

5. ELASTOPLASTIC DEFORMATION OF A SLOPE

Consider a plane slope of uniform soil with specific gravity γ_s. This slope is situated at an angle β to the horizontal, and its surface is unloaded. We will obtain the stress-strain state of this slope in the gravity field. We will introduce a system of coordinates xyz so that the y axis is directed into the slope while the $y = 0$ plane corresponds to its surface. The x axis is directed downwards along the surface of the slope while the z axis is directed transversely.

Since the slope is infinite in the x and z directions, all the physical quantities in this problem will be functions solely of the y coordinate, which enables us to seek the displacement vector in the form $\mathbf{u} = (u(y), \upsilon(y), 0)$. The stress and strain tensors will then have the following non-zero components: σ_x, σ_y, σ_z, τ_{xy} and $\gamma_{xy} = u'/2$, $\varepsilon_y = \upsilon'$ (the prime denotes a derivative with respect to y).

The equilibrium equations

$$\tau'_{xy} + \gamma_s \sin\beta = 0, \quad \sigma'_y + \gamma_s \cos\beta = 0$$

are easily integrated, and, taking the conditions $\sigma_y|_{y=0} = \tau_{xy}|_{y=0} = 0$ into account, we obtain

$$\tau_{xy} = -\gamma_s y \sin\beta, \quad \sigma_y = -\gamma_s y \cos\beta \tag{5.1}$$

In the elastic region the components $\sigma_x = \sigma_z$ are given by Hooke's law

$$\sigma_x = -\gamma_s y \frac{\nu}{1-\nu} \cos\beta \tag{5.2}$$

Then, we obtain the following expressions for the components of the strain tensor

$$\gamma_{xy} = -\frac{1}{2G}\gamma_s y \sin\beta = \frac{u'}{2}, \quad \varepsilon_y = -\frac{1}{2G}\frac{1-2\nu}{1-\nu}\gamma_s y \cos\beta = \upsilon' \tag{5.3}$$

The elastic solution obtained only holds in the region where the relation $\tau < (H - \sigma)\,\mathrm{tg}\alpha$ holds. Calculating the intensities of the shear stress τ and the mean pressure σ, we obtain the depth of the upper bound of the elastic deformation region

$$y_1 = \frac{H}{\gamma_s \cos\beta}\left(\mathrm{ctg}\alpha\sqrt{\mathrm{tg}^2\beta + \frac{1}{3}\xi^2} - \frac{1}{3}\eta\right)^{-1}; \quad \xi = \frac{1-2\nu}{1-\nu}, \quad \eta = \frac{1+\nu}{1-\nu} \tag{5.4}$$

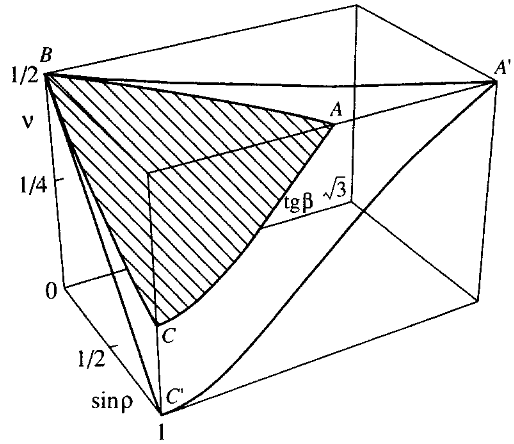

Fig. 4

It follows from formula (5.4) that if $\mathrm{tg}\alpha > \sqrt{3}\,\xi/\eta$ and the angle of the slope

$$\beta \le \beta_1 = \mathrm{arctg}(\sqrt{\eta^2 \mathrm{tg}^2\alpha - 3\xi^2/3}) \tag{5.5}$$

the whole slope will be in an elastic state and there will be no plastic region. This result will also be true for an ideally loose material, i.e. for a material with $H = 0$.

The surface in the parameter space of the problem α, β, ν, corresponding to the equality sign in formula (5.5), is shown in Fig. 4 for the case of a cone, inscribed into the Coulomb pyramid, i.e. tgα is given by the formula of (1.9) (the hatched region). The equations of the curves which bound this surface have the form

$$AB\colon\ \nu = \frac{1}{2},\ \ \mathrm{tg}\beta = \frac{\sqrt{3}\sin\rho}{\sqrt{3+\sin^2\rho}};\ \ AC\colon\ \sin\rho = 1,\ \ \nu = \frac{1+4\,\mathrm{tg}^2\beta}{5+4\,\mathrm{tg}^2\beta}$$

$$BC\colon\ \beta = 0,\ \ \sin\rho = \frac{1-2\nu}{\sqrt{\nu(2-\nu)}}$$

It is only possible for a plastic region to occur if the parameters of the problem lie within this surface. Its form depends very much on the approximation employed for the Coulomb criterion. The curves which bound the surface corresponding to the circumscribed cone, i.e. $\mathrm{tg}\alpha = 2\sqrt{3}\sin\rho/(3-\sin\rho)$, are represented by the heavy lines. The curve BC′: $\beta = 0$, $\sin\rho = 1 - 2\nu$, when the slope angle is equal to zero, is of particular intersect. In this case a stressed state of uniaxial compression with stresses σ_x arises in the slope (the base) with $\sigma_y = \sigma_z$, $\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$, and stressed states, which lead to the onset of plasticity, lie along the external faces of the Coulomb pyramid (modes A, C and E, see Fig. 1). In this case formula (5.4) can be obtained directly from formula (1.8) if we put $\sigma_2 = -\gamma_s y_1$, $\sigma_1 = \sigma_3 = \sigma_2\nu/(1-\nu)$ in the latter. The condition that, for such a compression of the medium, it remains in the elastic state, can be written as $\sin\rho \ge 1 - 2\nu$. For an ideally loose material ($H = 0$) strict inequality will denote that, for any uniaxial loading, the material remains elastic for any values of the compressing stress, and as $\sin\rho = 1 - 2$ plastic yield occurs.

If the angle $\beta > \beta_1$, a region $0 \le y < y_1$ of elastic deformation will exist in the soil, and the soil will be in the limit state at greater depths. Formulae (5.1) also remain true in the plastic region, but relation (5.2) and (5.3) turn out to be different, since it will be necessary to use the plastic-yield law (1.6), (1.7) instead of Hooke's law.

We will seek σ_x in the form

$$\sigma_x = -\gamma_s y(\cos\beta - \mu\sin\beta) \tag{5.6}$$

where $\mu = \mu(y)$ is an unknown factor. Relation (1.7) gives

$$\sqrt{\tau_{xy}^2 + (\sigma_x - \sigma_y)^2/3} = (H - (2\sigma_x + \sigma_y)/3)\,\mathrm{tg}\alpha$$

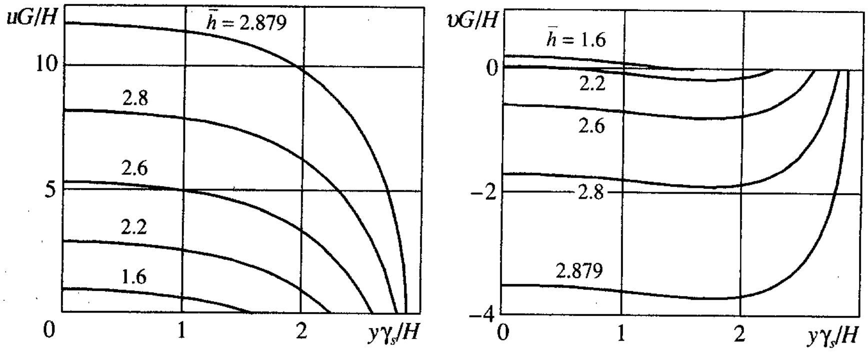

Fig. 5

or

$$(3-4\,\mathrm{tg}^2\alpha)\mu^2+12\mu\theta\,\mathrm{tg}^2\alpha+9(1-\theta^2\mathrm{tg}^2\alpha) = 0;\ \theta = \theta(y) = (H/(\gamma_s y)+\cos\beta)/\sin\beta$$

Solving this quadratic equation and substituting instead of tgα its expression in terms of the angle ρ (ctgα = $\sqrt{\mathrm{ctg}^2\rho + 4/3}$, the first formula in (1.9), the inscribed cone), we obtain

$$\mu = -2\theta\,\mathrm{tg}^2\rho+\sqrt{(\theta^2\mathrm{tg}^2\rho-1)(3+4\,\mathrm{tg}^2\rho)} \tag{5.7}$$

(the second root of the quadratic equation is discarded since it does not satisfy the condition of continuity of the stress tensor on the boundary between the elastic and plastic regions).

Knowing the components of the stress tensor, the strain tensor can be obtained from (1.6). After calculations we obtain

$$\begin{Vmatrix}\gamma_{xy}\\ \varepsilon_y\end{Vmatrix} = -\frac{H}{2G}\frac{3+4\,\mathrm{tg}^2\rho}{\theta-\mathrm{ctg}\beta}\left[\frac{\xi\eta^{-1}\mathrm{ctg}\beta+(2\eta^{-1}\mathrm{tg}^2\rho)\theta}{\sqrt{(\theta^2\mathrm{tg}^2\rho-1)(3+4\,\mathrm{tg}^2\rho)}}-\frac{2}{3}\left(\frac{\xi}{\eta}+\frac{2\,\mathrm{tg}^2\rho}{3+4\,\mathrm{tg}^2\rho}\right)\right]\begin{Vmatrix}1\\ \mu\end{Vmatrix} \tag{5.8}$$

The displacement field $u(y)$, $\upsilon(y)$ of the slope can be obtained by integrating the components of the strain tensor in accordance with the equation $u' = 2\gamma_{xy}$, $\upsilon' = \varepsilon_y$ with initial conditions $u(h) = 0$ and $\upsilon(h) = 0$, corresponding to a layer of depth h on a rigid base.

Analysis of expression (5.8) shows that as θ tends to ctgρ, which corresponds to depths

$$y \to y_2 = \frac{H}{\gamma_s}\frac{\sin\rho}{\sin(\beta-\rho)} \tag{5.9}$$

the deformations ε_y and γ_{xy}, and also the displacements u and υ, increase without limit, whereas the stresses σ_x, s_y and τ_{xy} remain finite. If $h = y_2$, the displacements on the lower boundary of the plastic layer reach infinitely high values, which physically corresponds to fracture of the slope.

For angles $\beta_1 \le \beta \le \rho$ a plastic region will exist in the soil, extending from y_1 to infinity (i.e. h can be as large as desired). If the slope angle is greater than ρ, the slope will be in equilibrium only if its depth does not exceed the critical value (5.9). Hence, when $\beta > \rho$ the plastic region can only occupy the layer $y \in [y_1, y_2]$.

The results of numerical calculations of the displacements $u(y)$ and $\upsilon(y)$ (in units of G/H) are presented in Fig. 5 for different values of the dimensionless depths $h = \gamma_s/H$ for $\rho = 30°$, $\beta = 40°$, and $\nu = 1/3$, to which the critical values $\beta_1 = 7.9°$, $y_0 = 1.75\,H/\gamma_s$, $y_1 = 1.10H/\gamma_s$, $y_2 = 2.8794\,H/\gamma_s$ correspond.

The problem can also be solved in a rigid-plastic formulation, i.e. as a limit equilibrium problem. The components of the stress tensor in the plastic region in this case also have the form (5.1). In the elastic layer the stress and strain states are not defined, and its weight must be regarded as an additional load acting on the boundary of the layer, which is in the limit state. The value of this load is determined by the height of the rigid layer y_1, which can be found from formula (5.4). For a given value of the slope angle β and the mechanical characteristics of the medium, α and H, the value of y_1 depends on Poisson's ratio ν and reaches its maximum value

$$y_0 = \frac{H}{\gamma_s}\frac{\sin\alpha}{\sin(\beta-\alpha)} \tag{5.10}$$

which corresponds to $\nu = 1/2$.

For a slope angle β satisfying the inequality $\alpha \le \beta \le \rho$, the value of h can be chosen arbitrarily in the range $y_0 < h < \infty$. Equations (2.16) for finding the displacement field are homogeneous in the components of the displacement, and for the boundary conditions $u(h) = \upsilon(h) = 0$, the components of the displacement u and υ will be identically equal to zero in the whole plastic region; the components of the stress tensor (5.1) and (5.6) in this region satisfy the yield criterion.

When the angle β becomes greater than ρ we must take $h = y_2$, and when $y \to y_2$ the angle φ, defined by the equation $\mathrm{tg}\,2\varphi = 2\tau_{xy}/(\sigma_x - \sigma_y)$, tends to the value $\varphi_0 = -\pi/4 - \rho/2$, i.e. the line $y = h$ is the envelope of the characteristics of the second family of (2.15), which may be lines of discontinuity of the displacements [3]. The values of one of the components can then be specified arbitrarily, while the second is found from the condition $\upsilon/u = \mathrm{tg}\,\rho$, i.e. in the plastic regions the components of the velocity are constant, while on the rigid-plastic boundary $y = y_2$ they suffer a discontinuity. Consequently, plastic yield of a massive slope of rigid plastic material is impossible at slope angles less than the angle of internal friction, although a solution in stresses exists. For slope angles greater than the angles of internal friction, only the part of the slope with a depth y_2 given by (5.8) can transfer into a state of plastic yield, not the whole layer. Despite the large number of publications devoted to solving this problem 1[1, 18], this result has been obtained here for the first time, and the elastoplastic solution described has played an important part which, as can easily be seen, transfers asymptotically into the solution corresponding to the theory of limit equilibrium.

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REFERENCES

1. SOKOLOVSKII, V. V., *The Statics of Granular Media*. Fizmatgiz, Moscow, 1960.
2. DRUCKER, D. C. and PRAGUER, W., Soil mechanics and plastic analysis or limit design. *Quart. App. Math.* 1952, **10**, 2, 157–165.
3. SHIRKO, I. V., Velocity fields under plasticity conditions of general form. *Trudy MFTI, Oborongiz, Moscow*, 1961, **7**, 71–84.
4. SHIRKO, I. V., Discontinuities of the stress fields under plasticity conditions of general form. *Izv. Akad. Nauk. Inzh. Zh.*, 1961, **1**, 3, 188–192.
5. SHIRKO, I. V., Some problems of the theory of plasticity with mixed boundary conditions. *Izv. Akad. Nauk SSSR. Inzh. Zh.*, 1962, **2**, 2, 305–311.
6. SHIRKO, I. V., *Effective Method of Numerical Solving the Problems of Plastic Flow of Materials*. Vol. 1. Proc. Adv. Tech. of Plasticity. Kyoto, 1990, 435–439.
7. DOLMATOV, B. I., BRONIN, V. N., KARLOV, V. D. *et al.*, *Soil Mechanics*. Pt 1, *Principles of Geotechnics in Building*. Izd. ASV, Moscow; SpbGA-SU, St Petersburg, 2000.
8. UKHOV, S. B., SEMENOV, V. V., ZNAMENSKII, V. V. *et al.*, *Soil Mechanics, Bases and Foundations*. Izd. ASV, Moscow, 1994.
9. SHIRKO, I. V., Elastoplastic bending of a plate clamped along the contour. *Trudy MFTI, Oborongiz, Moscow*, 1959, **3**, 180–186.
10. SHIRKO, I. V., Non-linear deformation and limit equilibrium of spatially curvilinear rods. *Prikl. Mat. Mekh.*, 1984, **48**, 1, 50–57.
11. SOKOLOVSKII, V. V., *The Theory of Plasticity*. Vyssh. Shkola, Moscow, 1969.
12. IL'YUSHIN, A. A., *Plasticity*. Gostekhizdat, Moscow, 1948.
13. KONDRAT'YEV, D. S. and SHIRKO, I. V., Simple deformation in the mechanics of granular media. In *Present Problems of Fundamental and Applied Sciences. Proceedings of the 43rd Conference*. MFTI, Moscow, 2000, Pt. 3.
14. SHIRKO, I. V., KONDRATYEV, D. S. and STETSENKO, P. V., Deformation theory in the mechanics of granular media. *Proc. American Soc. Composites: 17th Techn. Conf., West Lafayette*, 2002, No. 168.
15. NADAI, A., *Theory of Flow and Fracture of Solids*, Vol. 1. McGraw-Hill, New York, 1950.
16. BYKOV, D. L., Some methods of solving problems of the theory of plasticity. In *Elasticity and Inelasticity*. Izd. MGU, Moscow, 1975, No. 4, 119–139.
17. BURAGO, N. G. and KUKUDZHANOV, V. N., The solution of elastoplastic problems by the finite element method. In *Computational Mechanics of a Deformable Solid*. Nauka, Moscow, 1991, No. 2, 78–122.
18. NADAI, A., *Theory of Flow and Fracture of Solids*, Vol. 2. McGraw-Hill, New York, 1963.

19. SOKOLOVSKII, V. V., The equations of the theory of plasticity. *Prikl. Mat. Mekh.* 1955, **19**, 1, 41–45.
20. SHIRKO, I. V., KONDRAT'YEV, D. S. and STETSENKO, P. V., Deformation theory in the mechanics of granular media. In *The Mechanics of Composite Materials in Cosntructions*, 2002, **8**, 4, 555–573.
21. KHRISTIANOVICH, S. A., ZHELTOV, Yu. P., BARENBLATT, G. I. and MAKSIMOVICH, G. K., *Theoretical Principles of the Hydraulic Fracture of Oil Strata.* Selected Papers, Nauka, Moscow, 1998.
22. BURAGO, N. G. and KOVSHOV, A. N., The stressed state of rocks around a borehole. *Izv. Ross. Akad. Nauk. MTT,* 1999, 1, 139–143.

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